SOME FAMILIES OF EXACT SOLUTIONS OF THE EQUATIONS OF TWO-DIMENSIONAL SHALLOW WATER THEORY

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Results are presented of a study of the exact solutions of the equations of two-dimensional unsteady and steady shallow water theory, based on the group properties of these equations. The first part presents the group properties of the equations in question; the second part presents the invariant solutions of these equations.

We consider the equations of two-dimensional unsteady open liquid flows (shallow water theory or long wave theory)

$$D_t z + z \operatorname{div} \mathbf{u} = 0, \quad D_t \mathbf{u} + \nabla z = 0 \tag{0.1}$$

Here $\mathbf{u} = \{u_1, u_2\}$ is the velocity vector in the horizontal plane (x^1, x^2); $z = gh$, g is the gravity force acceleration, h is the stream depth; and t is time. The vector operator ∇ is the gradient operator with respect to the variable $\mathbf{x} = \{x^1, x^2\}$, div is the divergence symbol with respect to the variable $\mathbf{x}$, and the operator D_t is given by the formula

$$D_t = \frac{\partial}{\partial t} + u_1 \frac{\partial}{\partial x^1} + u_2 \frac{\partial}{\partial x^2}$$

The first equation (0.1) is the continuity equation and the second is the equation of motion. Equations (0.1) were first obtained by Friedrichs and Keller [1] in 1948 as the first approximation from the exact equations of hydrodynamics.

In many problems associated with river regulation, flood prediction, and so on, the nonlinear differential equations (0.1) with suitable additions, which make it possible to take into account the variation of the river valley bottom elevation and the flow resistance caused by irregularity of the river bottom, are used to study flows in rivers. In this case we have in place of the second equation (0.1)

$$D_t \mathbf{u} + \nabla z = -\nabla z_0 - F \mathbf{u} |\mathbf{u}| \tag{0.2}$$

Here $z_0 = z_0(x^1, x^2)$ is the bottom elevation; and $F = F(x^1, x^2, z)$ is the friction coefficient.

1. GROUP PROPERTIES

The objective in studying the group properties of differential equations lies in finding the broadest local Lie transformation group admitted by the given system of equations and in finding the classification of the invariant and partially invariant solutions of this system, introduced by Ovsyannikov [2].

There is a well-known technique for finding the infinitesimal operators of the given system, which leads to the solution of the defining equations for the coordinates of these operators [2]. We shall summarize the results, calculated using this technique, concerning the group properties of the system (0.1), (0.2), and the equations emanating from this system.

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For the equations (0.1), (0.2) the narrowest group, consisting of the single operator $X = \partial(\cdot)/\partial t$, occurs for arbitrary functions z_0 and F. Expansion of the group is permitted for special forms of z_0 and F which are not interesting from the physical viewpoint. This question was studied in [3] for the one-dimensional equations (0.1), (0.2). The broadest group occurs for z_0=const and F=0, i.e., for (0.1). The Lie algebra of the principal group admitted by (0.1) is generated by the linearly independent operators

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x^1}, \quad X_3 = \frac{\partial}{\partial x^2}, \quad X_4 = t\frac{\partial}{\partial x^1} + \frac{\partial}{\partial u_1} \tag{1.1}$$
$$X_5 = t\frac{\partial}{\partial x^2} + \frac{\partial}{\partial u_2}, \quad X_6 = t\frac{\partial}{\partial t} + x^1\frac{\partial}{\partial x^1} + x^2\frac{\partial}{\partial x^2}$$
$$X_7 = x^2\frac{\partial}{\partial x^1} - x^1\frac{\partial}{\partial x^2} + u_2\frac{\partial}{\partial u_1} - u_1\frac{\partial}{\partial u_2}, \quad X_8 = t\frac{\partial}{\partial t} - u_1\frac{\partial}{\partial u_1} - u_2\frac{\partial}{\partial u_2} - 2z\frac{\partial}{\partial z}$$
$$X_9 = t^2\frac{\partial}{\partial t} + tx^1\frac{\partial}{\partial x^1} + tx^2\frac{\partial}{\partial x^2} + (x^1 - tu_1)\frac{\partial}{\partial u_1} + (x^2 - tu_2)\frac{\partial}{\partial u_2} - 2tz\frac{\partial}{\partial z}$$

The system of three equations (0.1) will be a particular case of the system of four equations of two-dimensional unsteady gasdynamics [2]. Specifically, for A=2p and $p = \frac{1}{2}\rho^2$, the latter become the equations (0.1) (with replacement of ρ by z). In the case of the gasdynamic equations for A=2p there appears the additional operator

$$X_{10} = p\frac{\partial}{\partial p} + \rho\frac{\partial}{\partial \rho};$$

the sign changes in the last term of the expression for X_8 and the term $4tp\partial(\cdot)/\partial p$ appears in the expression for X_9.

The solutions of (0.1) will be partly invariant solutions of the equations of gasdynamics when the entropy S=const. It is well known that broadening of the group is possible in this case. It can be shown that the group admitted by the equations of isentropic motion follows from the group admitted by the equations of nonisentropic motion.

With account for the noted differences, the classification made in [4] of the invariant solutions of the equations of gasdynamics is valid for (0.1).

In addition to the invariant solutions, there are other classes of solutions which have received very little study to date. These include the partly invariant solutions of the simple wave and double wave types [5-7]. In studying double waves, the key equation [6] will be

$$(z - z_{u_2}^2) z_{u_1u_1} + 2z_{u_1}z_{u_2}z_{u_1u_2} + (z - z_{u_1}^2) z_{u_2u_2} + 2z - z_{u_1}^2 - z_{u_2}^2 = 0 \tag{1.2}$$

which is obtained from (0.1) under the assumptions

$$z = z(u_1, u_2), \qquad \frac{\partial u_2}{\partial x^1} - \frac{\partial u_1}{\partial x^2} = 0$$

Calculations of the group admitted by (1.2) yield the following operators:

$$X_1 = \frac{\partial}{\partial u_1}, \quad X_2 = \frac{\partial}{\partial u_2}, \quad X_3 = u_2\frac{\partial}{\partial u_1} - u_1\frac{\partial}{\partial u_2}, \quad X_4 = u_1\frac{\partial}{\partial u_1} + u_2\frac{\partial}{\partial u_2} + 2z\frac{\partial}{\partial z} \tag{1.3}$$

We note that the solution of the problem of the group properties of the following quasilinear system which is equivalent to (1.2) leads to the same result:

$$z_{u_1} - \tau = 0, \quad z_{u_2} - \sigma = 0, \quad \sigma_{u_1} - \tau_{u_2} = 0$$
$$(z - \sigma^2)\tau_{u_1} + 2\tau\sigma\tau_{u_2} + (z - \tau^2)\sigma_{u_2} + 2z - \tau^2 - \sigma^2 = 0$$

which is obtained by introducing the auxiliary variables τ and σ.

Using the internal automorphisms of the transformation group (1.3), we construct the optimal system of single-parameter subgroups

$$X_1, \ X_3 + \alpha X_4, \ X_4 \tag{1.4}$$

The invariant solutions of rank 2 relative to the dilational operator X_6 from (1.1) are called conical flows and have the form [8]

$$u_1 = x + u(x, y), \ u_2 = y + v(x, y), \quad z = h(x, y) \ (x = t^{-1}x^1, \ y = t^{-1}x^2) \tag{1.5}$$

Substituting (1.5) into (0.1), we obtain the system

$$uh_x + vh_y + h(u_x + v_y) = -2h$$
$$uu_x + vu_y + h_x = -u, \; uv_x + vv_y + h_y = -v \tag{1.6}$$

Calculations of the group admitted by the system (1.6) lead to the following results:

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial y}, \quad X_3 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} + v\frac{\partial}{\partial u} - u\frac{\partial}{\partial v}$$
$$X_4 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v} + 2h\frac{\partial}{\partial h} \tag{1.7}$$

The optimal system of single-parameter subgroups of the group (1.7) coincides with (1.4).

In the steady motion case [9], (0.1) take the form

$$u_1 z_{x^1} + u_2 z_{x^2} + z(u_{1x^1} + u_{2x^2}) = 0$$
$$u_1 u_{1x^1} + u_2 u_{1x^2} + z_{x^1} = 0, \quad u_1 u_{2x^1} + u_2 u_{2x^2} + z_{x^2} = 0, \tag{1.8}$$

which admit the group

$$X_1 = \frac{\partial}{\partial x^1}, \quad X_2 = \frac{\partial}{\partial x^2}, \quad X_4 = x^2\frac{\partial}{\partial x^1} - x^1\frac{\partial}{\partial x^2} + u_2\frac{\partial}{\partial u_1} - u_1\frac{\partial}{\partial u_2}$$
$$X_3 = x^1\frac{\partial}{\partial x^1} + x^2\frac{\partial}{\partial x^2}, \quad X_5 = u_1\frac{\partial}{\partial u_1} + u_2\frac{\partial}{\partial u_2} + 2z\frac{\partial}{\partial z} \tag{1.9}$$

The optimal system of single-parameter subgroups of the group (1.9) is given by the operators

$$X_1 + \beta X_5, \quad X_3 + \alpha X_4 + \beta X_5, \quad X_4 + \beta X_5 \tag{1.10}$$

In conclusion we note that the one-dimensional equations for shallow water with cylindrical waves

$$\frac{\partial z}{\partial t} + u\frac{\partial z}{\partial r} + z\frac{\partial u}{\partial r} + \frac{zu}{r} = 0, \quad \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + \frac{\partial z}{\partial r} = 0 \tag{1.11}$$

admit the following group of transformations:

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = t\frac{\partial}{\partial t} + r\frac{\partial}{\partial r}, \quad X_3 = t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u} - 2z\frac{\partial}{\partial z}$$
$$X_4 = t^2\frac{\partial}{\partial t} + tr\frac{\partial}{\partial r} + (r - tu)\frac{\partial}{\partial u} - 2tz\frac{\partial}{\partial z} \tag{1.12}$$

2. APPLICATIONS

In this section we examine the exact solutions of the shallow water equations and the equations which follow from them when using the group property results described in the first section.

A. Examples of Invariant Solutions of (0.1), (0.2). Knowledge of the basic group and optimal systems of one- and two-parameter subgroups permits finding the invariant solutions of rank 2 and 1, respectively. The optimal system of one-parameter subgroups of the group (1.1) contains 13 operators [4]; correspondingly, we have 13 invariant solutions of rank 2. The initial quantities will be functions of two arguments whose values are different for the different subgroups. The optimal system of two-parameter subgroups of the group (1.1) yields 30 invariant solutions of rank 1 when the initial quantities depend on a single argument. In both cases the solution of the systems of equations can be sought using numerical methods.

We shall note several examples of invariant solutions for the system (0.1), (0.2), which are written in explicit form. These solutions are of independent interest and may also be used to verify the numerical methods.

We start with the invariant solutions of rank 1.

a) We examine the invariant solution corresponding to the subgroup $H = \langle X_7, X_1 - X_6 + X_8 \rangle$. It has the form

$$u_r = rU, \quad u_\theta = rV, \quad z = r^2 H, \quad \lambda = re^t \tag{2.1}$$

Here r, θ are the polar coordinates in the plane x^1, x^2, and u_r, u_θ are the projections of the velocity on the polar coordinate axes.

We shall examine the motion of the water above a bottom whose equation is $z_0 = -\frac{1}{2}r^2$. without account for friction ($F = 0$).

Substituting (2.1) into (0.1), (0.2), written in polar coordinates, we find one of the solutions

$$U = 1, \ V = 0, \quad H = A\lambda^{-2},$$

where A is any positive number.

This yields the solution of the following boundary value problem. At the initial moment $t = 0$, there is given $u_r = r$, $u_\theta = 0$, $z = A (0 \le r \le a)$, and the boundary condition for $r = a (t > 0)$ has the form $u_r = a$. The solution has the form

$$u_r = r, \quad u_\theta = 0, \quad z = A \ \exp(-2t) \quad (0 \leqslant r \leqslant a, \ t \geqslant 0)$$

The bottom is not uncovered, although the depth decreases to zero with time.

b) For the subgroup $H = \langle X_6 - X_8, X_1 + X_9 \rangle$, we have the invariant solution

$$u_r = \frac{r(t+U)}{1+t^2}, \quad u_\theta = \frac{rV}{1+t^2}, \quad z = \frac{r^2 H}{(1+t^2)^2}, \quad \lambda = \theta \tag{2.2}$$

Substituting (2.2) into (0.1), we find one of the solutions

$$U = 0, \quad V = A, \quad H = \frac{1}{2}(A^2 - 1) \quad (A = \text{const}, \ A^2 > 1)$$

We pose the following problem: find the solution if at the initial moment $t = 0$, $u_r = 0$, $u_\theta = Ar$, $z = \frac{1}{2}(A^2 - 1)r^2$ $(0 \le r \le a)$, and for

$$r = a (t > 0) \quad u_r = at(1 + t^2)^{-1}$$

The solution has the form

$$u_r = \frac{rt}{1+t^2}, \quad u_\theta = \frac{Ar}{1+t^2}, \quad z = \frac{(A^2-1)r^2}{2(1+t^2)^2} \quad (0 \leqslant r \leqslant a, \ t \geqslant 0)$$

c) For the pair $\langle X_6, X_7 \rangle$ we have

$$u_r = U, \quad u_\theta = V, \quad z = H, \quad \lambda = rt^{-1} \tag{2.3}$$

Substituting (2.3) into (0.1), we find the particular solution

$$U = \frac{1}{2}\lambda, \quad V = A\lambda, \quad H = \frac{1}{8}(4A^2 + 1)\lambda^2$$

Let the conditions be given at the initial moment $t = 1$ $(0 \le r \le a)$

$$u_r = \frac{1}{2}r, \quad u_\theta = Ar, \quad z = \frac{1}{8}(4A^2 + 1)\ r^2$$

and at the boundary $r = a$ $(t > 1)$ $\mu_r = \frac{1}{2}at^{-1}$. The solution of this problem is given by the expressions

$$u_r = \frac{r}{2t}, \quad u_\theta = \frac{Ar}{t}, \quad z = \frac{4A^2+1}{8}\frac{r^2}{t^2} \quad (0 \leqslant r \leqslant a, \ t \geqslant 1)$$

This solution is qualitatively reminiscent of the preceding solution.

d) On the subgroup $H = \langle X_7, X_1 + X_9 \rangle$ we have the invariant solution

$$u_r = \frac{rt}{1+t^2} + \frac{U}{r}, \quad u_\theta = \frac{V}{r}, \quad z = \frac{H}{1+t^2}, \quad \lambda = \frac{r}{\sqrt{1+t^2}} \tag{2.4}$$

We consider the case in which the bottom is given by the equation

$$z_0 = \frac{1}{2} Ar^{-2} \ (A < 0)$$

We ignore friction ($F = 0$). Substituting (2.4) into (0.1) and (0.2), we find the particular solution

$$U = 0, \quad V = 0, \quad H = -\frac{1}{2} A\lambda^{-2} - \frac{1}{2}\lambda^2 + 2b^2$$

Let at the initial moment

$$t = 0 \ \ (a \leqslant r \leqslant \sqrt{2}[1 + (1 - \frac{1}{2} A / b^4)^{1/2}]^{1/2} \ b, \quad 0 < a < \sqrt{2}b)$$

given the state

$$u_r = 0, \quad u_\theta = 0, \quad z = 2b^2 - {}^1/_2 r^{-2}(A + r^4),$$

and at the boundary r = a (t > 0)

$$u_r = \frac{at}{1+t^2}, \qquad z = \frac{2b^2}{1+t^2} - \frac{1}{2a^2}\left(A + \frac{a^4}{(1+t^2)^2}\right)$$

The solution of this problem is given by the expressions

$$u_r = \frac{rt}{1+t^2}, \quad u_\theta = 0, \quad z = \frac{2b^2}{1+t^2} - \frac{1}{2r^2}\left(A + \frac{r^4}{(1+t^2)^2}\right)$$
$$a \leqslant r \leqslant \sqrt{2}\,[1 + (1 - {}^1/_2 A / b^4)^{1/2}]^{1/2} (1 + t^2)^{1/2} b, \qquad t \geqslant 0$$

The liquid boundary (z = 0) travels following the law

$$r = \sqrt{2}[1 + (1 - {}^1/_2 A / b^4)^{1/2}]^{1/2}\, b(1 + t^2)^{1/2}$$

As t→∞ the level approaches a horizontal position.

A particular case of this solution, namely, when A = 0 (horizontal bottom), was studied in [4].

In the preceding solutions friction was ignored: we shall now present an example which accounts for friction.

e) On the subgroup H = $\langle X_7, X_6 + X_8 \rangle$ we obtain the invariant solution

$$u_r = r^{-1}U, \quad u_\theta = r^{-1}V, \quad z = r^{-2}H, \quad \lambda = r^{-2}t \tag{2.5}$$

We consider the case in which

$$z_0 = -\frac{A}{2r^2}\,(A > 0), \qquad F = \frac{A}{2r^3 z}$$

Substituting (2.5) into (0.1), (0.2), we find one of the solutions

$$U = {}^1/_2\, \lambda^{-1}, \quad V = 0, \quad H = {}^1/_2 A$$

We pose the problem: find the solution if at the initial moment t = 1 ($a \leq r \leq b$) there is given

$$u_r = {}^1/_2\, r, \quad u_\theta = 0, \quad z = {}^1/_2 r^{-2} A,$$

and at the boundaries we have

$$u_r = {}^1/_2\, at^{-1}, \quad z = {}^1/_2\, Aa^{-2} \quad \text{for} \quad r = a(t > 1),$$
$$u_r = {}^1/_2\, bt^{-1} \quad \text{for} \quad r = b \quad (t > 1)$$

The solution of this problem is given by the expressions

$$u_r = {}^1/_2 t^{-1} r, \; u_\theta = 0, \quad z = {}^1/_2\, Ar^{-2} \qquad (a \leqslant r \leqslant b, \; t \geqslant 1)$$

It is interesting to note that the elevation is constant, since $z + z_0 \equiv 0$, i.e., at any time the free surface is horizontal and constant.

Let us turn to the invariant solutions of rank 2. The operator $X_1 + X_4$ corresponds to an invariant solution of the form

$$u_1 = t + U, \quad u_2 = V, \quad z = H, \quad \lambda = {}^1/_2 t^2 - x^1, \quad \mu = x^2 \tag{2.6}$$

We consider the case in which the bottom is an inclined plane

$$z_0 = -ix^1 - jx^2 \quad (i \neq 1, \; j \neq 0)$$

We neglect friction.

Substituting (2.6) into (0.1), (0.2), we obtain a system in partial derivatives with two independent variables, whose coefficients are independent of λ and μ. Therefore this system admits a solution of the simple wave type, i.e.,

$$U = U(\theta), \quad V = V(\theta), \quad H = H(\theta) \qquad (\theta = \theta_0 + a_1\lambda + a_2\mu)$$

One of the simple wave solutions has the form

$$U = 0, \quad V = 0, \quad H = (1 - i)a^{-1}\theta \quad (a = \text{const}, \quad a \neq 0)$$

This yields the solution of the following problem: at the initial moment $t = 0$ the liquid is at rest and its free surface level coincides with the plane

$$z = (1 - i)a^{-1}\theta_0 - (1 - i)x^1 + jx^2$$

Thereafter (for $t > 0$) the liquid motion is described by the expressions

$$u_1 = t, \quad u_2 = 0, \quad z = (1 - i)a^{-1}\theta_0 + (1 - i)(1/_2 t^2 - x^1) + jx^2$$

The liquid boundary ($z = 0$) travels with the velocity

$$|1 - i| [(1 - i)^2 + j^2]^{-1/_2}\, t$$

and the free surface displaces parallel to itself.

Let us examine the operator $X_1 + X_9$. It corresponds to an invariant solution of the form

$$u_1 = \frac{tx^1}{1 + t^2} + \frac{U}{x^1}, \quad u_2 = \frac{tx^2}{1 + t^2} + \frac{V}{x^2}, \quad z = \frac{H}{1 + t^2}, \quad \lambda = \frac{x^1}{\sqrt{1 + t^2}}, \quad \mu = \frac{x^2}{\sqrt{1 + t^2}} \tag{2.7}$$

Substituting (2.7) into (0.1), we obtain a system whose particular solution has the form

$$U = 0, \quad V = 0, \quad H = 1/_2\, (a^2 - \lambda^2 - \mu^2)$$

This is the solution found in [4] and noted above, which has the form

$$u_r = \frac{tr}{1 + t^2}, \quad u_\theta = 0, \quad z = \frac{1}{2(1 + t^2)}\left(a^2 - \frac{r^2}{1 + t^2}\right) \tag{2.8}$$

$$(0 \leqslant r \leqslant a\sqrt{1 + t^2}, \; t \geqslant 0)$$

Thus, the simplest invariant solutions which we have been able to find are described by the one-dimensional equations (1.11) of shallow water with cylindrical waves.

Knowledge of the basic group (1.12) makes it possible to find the invariant solutions of rank 1 of the system (1.11). For example, the operator $\alpha X_2 - (\alpha - 1) \times X_3$ corresponds to the self-similar solution

$$u = rt^{-1}U, \quad z = r^2 t^{-2} H, \quad \lambda = rt^{-\alpha}$$

and the solution (2.8) corresponds to the operator $X_1 + X_4$ from the group (1.12).

B. Simple Waves. Such waves are partly invariant solutions of rank 1 [5]. In this case

$$\mathbf{u} = \mathbf{u}(\xi), \quad z = z(\xi), \quad \xi = \xi(x^1, \; x^2, \; t) \tag{2.9}$$

We denote

$$\frac{d\xi}{dt} = \frac{\partial \xi}{\partial t} + u_1 \frac{\partial \xi}{\partial x^1} + u_2 \frac{\partial \xi}{\partial x^2}$$

a) Let

$$d\xi \,/\, dt = 0 \tag{2.10}$$

Equation (2.10) means that the level surfaces of the simple wave will be contact characteristics. From (0.1) with account for (2.9), (2.10), we obtain

$$z = \text{const}, \quad \mathbf{u}^{\cdot}\, \nabla \xi = 0$$

This is the case of a degenerate simple wave.

b) Let us examine the nondegenerate simple wave

$$d\xi \,/\, dt \neq 0$$

It is easy to show that the nondegenerate simple wave is an irrotational motion. In fact, multiplying the second equation (0.1) with account for (2.9) vectorially by $\mathbf{u}^{\cdot}$, we obtain

$$\mathbf{u}^{\cdot} \times \nabla \xi = 0 \tag{2.11}$$

Exclusion of $\mathbf{u}^{\cdot}$ from (0.1) leads to the relation

$$\left(\frac{d\xi}{dt}\right)^2 = c^2 |\nabla\xi|^2 \qquad (c^2 = z) \tag{2.12}$$

This equation shows that the level surfaces ξ = const of the nondegenerate simple wave will be sonic characteristics (term borrowed from gasdynamics). Exclusion of $\mathbf{u}^{\cdot}\nabla\xi$ leads to the relation

$$z\,(\mathbf{u}^{\cdot})^2 = (z^{\cdot})^2 \tag{2.13}$$

It also follows from (2.11) that the normal to the surface ξ = const has the same direction for all of its points. Therefore, each level surface of the simple wave is a plane in $E_3(x^1, x^2, t)$ space.

Hence we have

$$u_1^{\cdot} x^1 + u_2^{\cdot} x^2 - (qq^{\cdot} + z^{\cdot})t = P(\xi) \quad (q = |\mathbf{u}|) \tag{2.14}$$

Here $P(\xi)$ is an arbitrary function. The complete description of nondegenerate simple waves is given by (2.13) and (2.14). Thus, the simple waves of (0.1) have been studied completely.

C. Double Waves. These are partly invariant solutions of rank 2. The double wave concept was first introduced in [6], and its group nature was described in [7]. In this case

$$u = u(\xi, \eta), \quad z = z(\xi, \eta), \quad \xi = \xi(x^1, x^2, t), \; \eta = \eta\,(x^1, x^2, t) \tag{2.15}$$

The functions ξ and η are called the parameters of the double wave. They can be selected in several ways. We shall consider the unknowns u_1 and u_2 to be independent and take them as the parameters ξ and η. Then we obtain from (0.1), under the condition that the flow be potential, the equation (1.2), admitting the group (1.3).

For convenience in writing the invariant solutions, we convert in (1.2) to polar coordinates

$$q = \sqrt{u_1^2 + u_2^2}, \qquad \theta = \mathrm{arc\,tg}\,(u_2 / u_1)$$

Then (1.2) takes the form

$$\begin{gathered}(z - q^{-2}z_\theta^2)z_{qq} + 2q^{-2}z_q\, z_\theta\, z_{q\theta} + q^{-2}(z - z_q^2)z_{\theta\theta} - q^{-1}z_q^3 - 2q^{-3}z_q\, z_\theta^2 \\ - z_q^2 - q^{-2}z_\theta^2 + q^{-1}zz_q + 2z = 0,\end{gathered} \tag{2.16}$$

and the admissible operators (1.3) will be

$$\begin{gathered}X_1 = \cos\theta\frac{\partial}{\partial q} - \frac{\sin\theta}{q}\frac{\partial}{\partial\theta}, \qquad X_2 = \sin\theta\frac{\partial}{\partial q} + \frac{\cos\theta}{q}\frac{\partial}{\partial\theta} \\ X_3 = -\frac{\partial}{\partial\theta}, \qquad X_4 = q\frac{\partial}{\partial q} + 2z\frac{\partial}{\partial z}\end{gathered} \tag{2.17}$$

Using the optimal subgroups (1.4), it is easy to write out the corresponding essentially different invariant solutions.

a) We can construct an invariant solution corresponding to the operator $X_3 = -\partial(\cdot)/\partial\theta$. It has the form $z = z(q)$. In this case we obtain in place of (2.16) the second-order ordinary differential equation

$$qzz'' - z'^3 - qz'^2 + zz' + 2qz = 0 \tag{2.18}$$

Equation (2.18) admits reduction of the order. The substitutions

$$z(q) = q^2\eta\,(\xi), \; \xi = \ln q, \;\; p(\eta) = \eta'(\xi)$$

lead to the equation

$$pp' - p^3 - (1 - 6\eta)p^2 + 3(1 - \eta - 4\eta^2)p - 8\eta^3 - 2\eta^2 + 4\eta = 0$$

b) To the operator

$$X_4 = q\frac{\partial}{\partial q} + 2z\frac{\partial}{\partial z}$$

corresponds an invariant solution of the form

$$z = q^2 H(\theta) \tag{2.19}$$

Substituting (2.19) into (2.16), we obtain the ordinary differential equation

$$H(1-4H)H'' + (2H-1)H'^2 - 8H^3 + 2H = 0$$

which has the solution

$$\theta = \pm \int \frac{\sqrt{1-4H}\, dH}{\sqrt{H(CH-16H^2-4)}}$$

c) For the operator

$$X_3 + \alpha X_4 = \alpha q \frac{\partial}{\partial q} - \frac{\partial}{\partial \theta} + 2\alpha z \frac{\partial}{\partial z}$$

we have

$$z = q^2 H(\lambda), \qquad \lambda = q e^{\alpha\theta}$$

d) For the operator $X_1 = \partial(\cdot)/\partial u_1$ we have an invariant solution of the form $z = z(u_2)$. Substituting into (1.2), we obtain the equation $zz'' - z'^2 + 2z = 0$, which has the solution

$$u_2 = \pm \int \frac{dz}{\sqrt{z(C - 2\ln z)}}$$

We have a similar solution for the operator $X_2 = \partial(\cdot)/\partial u_2$.

Examples of double waves are one-dimensional flows with plane waves and two-dimensional steady-state flows. The complete classification of double waves has not yet been obtained.

D. Conical Flows. One class of particular solutions of (0.1) will be the conical flows – a particular case of double waves. The system (0.1) admits [see (1.1)] the dilatation operator

$$X_6 = t\frac{\partial}{\partial t} + x^1 \frac{\partial}{\partial x^1} + x^2 \frac{\partial}{\partial x^2}$$

The invariant solutions of rank 2 relative to this operator are termed conical flows. The invariants of this operator will be the quantities $x = t^{-1}x^1$, $y = t^{-1}x^2$; therefore, the conical flows have the form (1.5). Conical flows occur whenever the flow region is bounded by straight lines converging to a single point, which thereafter displace with constant velocity, causing the walls to remain wetted. As examples we note the following cases.

a) The angular piston problem, in which at the moment $t = 0$ the entire wall begins to move away from the liquid with constant velocity.

b) A discontinuous wave approaches the apex of a wedge at the moment $t = 0$.

Conical flows are described by (1.6). The system (1.6) will be remarkable in the sense that these are the equations (1.8) of two-dimensional steady motion with certain forces and sources. But the system (1.6) is more complex than the system (1.8), and even (1.8) has received very little study [9]. Finding the invariant solutions of (1.6) will reduce to integrating a system of ordinary differential equations. We note the invariant solutions of rank one of the system (1.6), whose basic group operators are given by (1.7), and the optimal system of single-parameter subgroups is given by (1.4).

a) For the operator $X_1 = \partial(\cdot)/\partial x$ we have the invariant solution of the form $u = U(y)$, $v = V(y)$, $h = H(y)$, which will be one-dimensional.

After determining $V(y)$ from the equation

$$(V' + 1)(3V'^2 + 8V' + 4) = CV^2,$$

$U(y)$ and $H(y)$ are found from the formulas

$$U(y) = \exp\left(-\int \frac{dy}{V(y)}\right), \qquad H(y) = \frac{V^2(V'+1)}{V'+2}$$

A similar solution holds for $X_2 = \partial(\cdot)/\partial y$.

b) For the operator X_4, which in polar coordinates has the form

$$r\frac{\partial}{\partial r}+u\frac{\partial}{\partial u}+v\frac{\partial}{\partial v}+2h\frac{\partial}{\partial h},$$

we have

$$u=rU(\theta),\quad v=rV(\theta),\quad h=r^2H(\theta)$$

c) For the operator X_3 in polar coordinates

$$-\frac{\partial}{\partial\theta}+v\frac{\partial}{\partial u}-u\frac{\partial}{\partial v},$$

we have

$$u=V\sin(U-\theta),\ v=V\cos(U-\theta),\quad h=H,\quad \lambda=r$$

d) For the operator $X_3+\alpha X_4$ in polar coordinates

$$\alpha r\frac{\partial}{\partial r}-\frac{\partial}{\partial\theta}+(\alpha u+v)\frac{\partial}{\partial u}+(\alpha v-u)\frac{\partial}{\partial v}+2\alpha h\frac{\partial}{\partial h}$$

we have

$$u=rV\sin(U-\theta),\quad v=rV\cos(U-\theta),\quad h=r^2H,\quad \lambda=re^{\alpha\theta}$$

E. Steady-State Flow. In the case of steady flow the basic equations will be (1.8), whose group operators are given by (1.9), and the optimal system of one-parameter subgroups is given by the operators (1.10).

We note the invariant solutions.

a) For the operator

$$X_1+\beta X_5=\frac{\partial}{\partial x^1}+\beta u_1\frac{\partial}{\partial u_1}+\beta u_2\frac{\partial}{\partial u_2}+2\beta z\frac{\partial}{\partial z}$$

we have

$$u_1=e^{\beta x^1}U(x^2),\qquad u_2=e^{\beta x^1}V(x^2),\qquad z=e^{2\beta x^1}H(x^2)$$

b) For the operator $X_4+\beta X_5$ we have

$$u_1=e^{\beta\theta}V\sin(U-\theta),\quad u_2=e^{\beta\theta}V\cos(U-\theta),\quad z=e^{2\beta\theta}H,\quad \lambda=r$$

c) For the operator $X_3+\alpha X_4+\beta X_5$ in polar coordinates

$$r\frac{\partial}{\partial r}-\alpha\frac{\partial}{\partial\theta}+(\beta u_1+\alpha u_2)\frac{\partial}{\partial u_1}+(\beta u_2-\alpha u_1)\frac{\partial}{\partial u_2}+2\beta z\frac{\partial}{\partial z}$$

we have

$$u_1=r^\beta V\sin(U-\theta),\quad u_2=r^\beta V\cos(U-\theta),\quad z=r^{2\beta}H,\quad \lambda=r^\alpha e^\theta$$

d) For the operator X_3 we have

$$u_1=U(\lambda),\quad u_2=V(\lambda),\quad z=H(\lambda),\quad \lambda=x^1/x^2$$

We obtain the constant solution $u_1=A$, $u_2=B$, $z=C$.

As an example, we consider the invariant solution corresponding to X_4 from (1.9). It has the form

$$u_r=U(r),\quad u_\theta=V(r),\quad z=H(r) \tag{2.20}$$

We shall examine water flow over a bottom specified in the form $z_0=z_0(r)$ without account for friction ($F=0$). Substituting (2.20) into (0.1) and (0.2), written in polar coordinates, we find one of the solutions

$$U=0,\quad V=V(r),\quad H=\int r^{-1}V^2dr-z_0,$$

where $V(r)$ is an arbitrary function.

Hence we see that the free surface elevation $H + z_0$ in this circular motion is independent of z_0, i.e., the motion is completely defined by specification of $V(r)$ and z_0 affects only the depth.

We can consider the partly invariant solutions of the equations of steady-state flow – simple waves, which have been studied in detail in [5]. The description of all the nondegenerate simple waves for steady-state flow follows from Eqs. (2.13) and (2.14) for unsteady flow. The properties of the characteristics and simple waves for steady supercritical liquid flows are the same as in the case of one-dimensional unsteady flows with plane waves.

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